

Self-organized criticality in glassy spin systems requires long-range interactions

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We investigate the conditions required for general spin systems with frustration and disorder to display self-organized criticality, a property which so far has been established only for the infinite-range Sherrington-Kirkpatrick Ising spin-glass model [Phys. Rev. Lett. **83**, 1034 (1999)]. Here we study both avalanche and magnetization jump distributions triggered by an external magnetic field, as well as internal field distributions in the short-range Edwards-Anderson Ising spin glass for various space dimensions between 2 and 8. Our numerical results, obtained on systems of unprecedented size, demonstrate that self-organized criticality is recovered only in the strict limit of infinite space dimensions (or equivalently of long-ranged interactions), and is not a generic property of spin-glass models in finite space dimensions.

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Self-organized criticality (SOC) refers to the tendency of large dissipative systems to drive themselves into a scale-invariant critical state without any special tuning of the parameters [1]. These phenomena are of crucial importance, because fractal objects displaying SOC are found everywhere [2]: one encounters them in earthquakes, in the structure of dried-out river beds, the meandering of sea coasts, or in the structure of galactic clusters. Understanding its origin, however, represents a major unresolved puzzle of modern science, because in most equilibrium systems critical behavior featuring scale-free (fractal) patterns is found only at isolated critical points, and is not a *generic* feature across phase diagrams.

Pioneering work in the 1980s provided important insights into the possible origin of SOC by identifying few theoretical examples that display it. The sand-pile model [3] and the forest-fire model [4] are hallmark examples of dynamical systems that exhibit SOC. However, these models feature *ad hoc* dynamical rules, without showing how these can be obtained from an underlying Hamiltonian dynamics. Major questions thus remain: First, can one obtain SOC from an underlying Hamiltonian model? Second, is this behavior a feature of high-dimensional models or models with long-range interactions, or is it a generic property of broad class of systems?

However, work in the 1990s offered a glint of hope. The first Hamiltonian model displaying SOC *without* the tuning of any parameters was identified and studied in detail in the pioneering work of Pazmandi *et al.* [5]. Here, the infinite-ranged Sherrington-Kirkpatrick (SK) spin-glass model [6] was examined; out-of-equilibrium avalanches at zero temperature ($T = 0$) triggered by varying the magnetic field were numerically studied along the hysteresis loop. A distinct power-law behavior in the distribution of spin avalanches, as well as of the magnetization jumps

was established, i.e., SOC.

The possible existence of SOC was also tested in several finite-dimensional models, *but in all these cases at least one parameter has to be tuned* for the system to display scale-free avalanches. The best-studied such model is the random-field Ising model where ferromagnetic Ising spins are coupled to a random field of average strength R . For space dimensions d larger than two, a critical value R_c exists where avalanches and magnetization jumps along a hysteresis loop show signs of criticality, i.e., the relevant distributions assume a power-law form [7–11]. We stress that in this model, tuning of the parameter R is needed; only at $R = R_c$ scale-free avalanches occur. Similar results were found for the random-bond Ising model [12] where spins interact with bonds J_{ij} drawn from a Normal distribution with a ferromagnetic mean unity and standard deviation σ . For a critical value of $\sigma_c \approx 0.44$ signs of criticality are found. Finally, studies on the random anisotropy Ising model [13] also showed a similar behavior when the system is tuned to criticality.

Surprisingly, no numerical studies have been reported to date for the “vanilla” Edwards-Anderson Ising spin glass (EASG) [14] (Normal-distributed random interactions with zero mean). Recently, Le Doussal *et al.* [15] suggested the possibility of SOC in the EASG for finite space dimensions. Their calculation is based on droplet arguments and suggests that at least for fields close to zero SOC should be present when studying avalanches in the EASG. Although their considerations are done for systems in equilibrium, their results suggest that SOC might be present in out-of-equilibrium avalanche simulations as done for the SK model in Ref. [5].

A deeper understanding of models that exhibit SOC is thus needed. Because the SK model is thought to be the mean-field limit of the EASG, standard lore would

suggest that the short-range EASG may display SOC for all space dimensions $d \geq 6$, i.e., above the upper critical dimension d_u where mean-field behavior sets in. To understand whether mean-field behavior suffices, or truly long/infinite-range interactions are needed, we study field-driven avalanche distributions at zero temperature for the EASG in $d = 2, 3, 4, 6$, and 8 space dimensions. Therefore, we probe the system below, at, and above the (equilibrium) upper critical dimension. In addition, we compare to results for $d = \infty$, i.e., the SK model. Our results, obtained for extremely large system sizes, demonstrate that as long as the space dimension $d < \infty$ no SOC is present in the EASG. However, for $d = \infty$ (SK model), SOC is recovered. Our results therefore indicate that long-range interactions represent the key ingredient to obtain parameter-free avalanches and SOC in spin systems.

Model, Observables and Algorithm.— We study glassy Ising models in d space dimensions with the Hamiltonian

$$\mathcal{H} = - \sum_{i < j}^N J_{ij} S_i S_j - H \sum_i S_i. \quad (1)$$

Here $S_i = \pm 1$ represent $N = L^d$ Ising spins on a hypercubic lattice of linear size L . The interactions J_{ij} between the spins are drawn from a Normal distribution with zero mean, and H represents an external magnetic field used to drive the avalanches. For $d = \infty$, i.e., the SK spin glass [6], the sum is over all spins and the variance of the spin-spin interactions is chosen as $1/(N-1)$ to ensure extensive quantities. When $d < \infty$ the model is known as the Edwards-Anderson Ising spin glass [14], where the interactions are limited to nearest neighbors and have variance unity.

The numerical method used is zero-temperature Glauber dynamics, which has been successfully used before [7, 9, 16]. We start by computing the local fields for all spins: $h_i = \sum_j J_{ij} S_j - H$. A spin is unstable if the stability $h_i S_i < 0$. The initial field H is selected to be larger than the largest local field, i.e., $H > |h_i| \forall i$. The spins are then sorted by local fields and the field H reduced until the stability of the first sorted spins crosses zero, making the spin unstable [17]. This unstable spin is flipped, then the local fields/stabilities of the other spins are recalculated, and the most unstable spin is flipped again. The process is repeated until all spins become stable, i.e., their stabilities are nonnegative. In most cases, the flipping of the first unstable spin triggers the flipping of a substantial number of other spins, therefore causing an avalanche. A focus of the present paper is to characterize the statistics of these avalanches. Simulation parameters are shown in Table I.

At each avalanche triggered by the above algorithm, we measure the number of spins n that flipped until the system regains equilibrium and record the distribution

TABLE I: Parameters of the simulation: For each space dimension d we study systems of $N = L^d$ spins ($d < \infty$) and compute the different distributions over N_{sa} disorder samples.

d	L	N	N_{sa}
2	1000	1 000 000	15 000
2	2000	4 000 000	15 000
2	3000	9 000 000	14 880
2	4000	16 000 000	14 860
3	100	1 000 000	15 000
3	150	3 375 000	10 000
3	200	8 000 000	12 900
3	250	15 625 000	14 250
4	10	10 000	15 000
4	20	160 000	15 000
4	40	2 560 000	15 000
4	60	12 960 000	15 000
6	8	262 144	15 000
6	10	1 000 000	15 000
6	12	2 985 984	15 000
6	14	7 529 536	15 000
6	16	16 777 216	15 000
8	4	65 536	15 000
8	5	390 625	15 000
8	6	1 679 616	15 000
8	7	5 764 801	14 480
8	8	16 777 216	10 200
∞	N/A	2 500	100 000
∞	N/A	5 000	19 800
∞	N/A	10 000	14 900
∞	N/A	20 000	14 860
∞	N/A	30 000	19 795

of avalanche sizes $D(n)$ for all triggered avalanches until $S_i \rightarrow -S_i \forall i$. In addition, we measure the magnetization jump S at each avalanche and record the distribution of magnetization jumps $P(S)$ [18]. For the SK model the avalanches are expected to be power-law distributed with an exponential cutoff that sets in at a characteristic size n^* (similar arguments are valid for the magnetization jumps with a characteristic size S^*). Only if $n^*(N) \rightarrow \infty$ as $N \rightarrow \infty$ does the system exhibit true SOC. We determine n^* in two different ways: First, we fit the tail of the distributions to $D(n) \sim \exp[-n/n^*(N)]$ with $n^*(N)$ an fitting parameter. We also fit the small- n regime to a power-law and determine the point of closest proximity between the power-law and exponential cutoff fits. This yields a second estimate of $n_c^*(N)$. While the $n^*(N)$ values obtained by the two approaches can differ by as much as a factor of ~ 2 , $n^*(N \rightarrow \infty)$ obtained by either definition exhibits the same qualitative behavior. In the simulations we fit the distributions and extract $n^*(N)$ for a given space dimension d and (linearly) extrapolate to $N = \infty$.

In addition, to study criticality for $H \sim 0$ in the short-range systems, we measure the avalanche distribution $D_0(n)$ and magnetization jump distribution $P_0(S)$ if and only if the field H crosses zero. These measurements are necessary for short-range systems because the existence of a spin-glass state in a field remains controversial [19–24]. Therefore, by restricting the measurements to $H = 0$ we expect to probe an actual (non-equilibrium) spin-glass state.

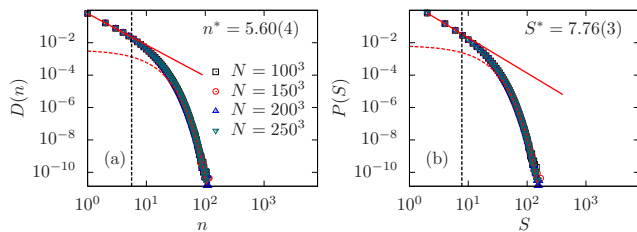


FIG. 1: (Color online) (a) Avalanche distribution $D(n)$ for the Edwards-Anderson spin-glass model in $d = 3$ recorded across the whole hysteresis loop. (b) Magnetization jump distribution $P(S)$ recorded across the whole hysteresis loop. For both quantities, the data show no finite-size effects. The solid line represents a power-law fit, whereas the dashed curve represents an exponential fit (see text). The vertical (black) dashed line marks the crossover value where a power-law behavior changes into an exponential cutoff (n^* and S^* are determined by a fit to an exponential cutoff function, see text).

Ref. [5] argued that a true SOC system suppresses avalanche formation and stabilizes itself by developing a power-law pseudo-gap in $P(h)$, the distribution of stabilities, similar to the Efros-Shklovski gap of the Coulomb glass. Requiring the system to be stable against avalanches gives stringent bounds on the exponent and the coefficient of the power-law form. Therefore, we also study the distribution of local fields (stabilities) $P(h)$ for h close to zero.

Results.— Figure 1 shows data for the avalanche size and magnetization jump distributions for the Edwards-Anderson spin glass in three space dimensions ($d = 3$). The data show that avalanches remains small, even for systems with millions of spins, i.e., the cutoffs n^* and S^* do not scale with the system size. In fact, even though we simulate over 10^7 spins, the largest avalanches (which occur extremely rarely) are only of approximately 100 spins. A crossover from a power-law to an exponential cutoff in the distributions occurs for rather small values of n and S , respectively, suggesting no SOC. The vertical dashed line in Fig. 1 corresponds to the extrapolated crossover values for n^* and S^* , respectively.

Reference [5] reported that the SK model—typically viewed as the mean-field limit/formulation of the EA model—exhibits SOC. Therefore, one could expect that the EA model exhibits SOC above its upper critical dimension $d_u = 6$. To test this expectation, we simulate systems in $d = 8$ space dimensions. Again, no visible system-size dependence or power-law behavior are present, indicating that the system displays no SOC. To sidestep the debate over the existence of a spin-glass state in a field, we measure the avalanches only when H crosses zero, i.e., where it is most probable that the system is in a spin-glass state, even if a nonequilibrium-type of spin-glass state [18]. Because fluctuations are large when the restricted magnetization is measured the data are noisy. However, again no large avalanches or magnetiza-

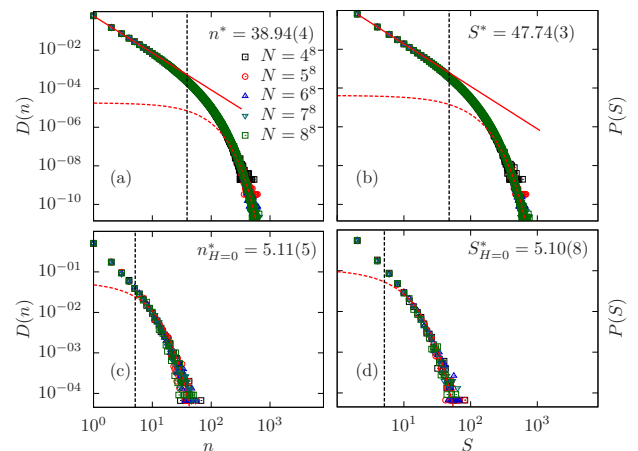


FIG. 2: (Color online) (a) Avalanche distribution $D(n)$ for the Edwards-Anderson spin-glass model in $d = 8$ recorded across the whole hysteresis loop. (b) Magnetization jump distribution $P(S)$ recorded across the whole hysteresis loop. (c) Avalanche distribution $D_0(n)$ restricted to $H = 0$. (d) Magnetization jump distribution $P_0(S)$ restricted to $H = 0$. As in Fig. 1 the data show no finite-size effects. The solid line represents a power-law fit, whereas the dashed curve represents an exponential fit (see text). The vertical (black) dashed lines mark the crossover value where a power-law behavior changes into an exponential cutoff (n^* and S^* are determined by a fit to this exponential cutoff, see text). Panels have matching vertical and horizontal scales.

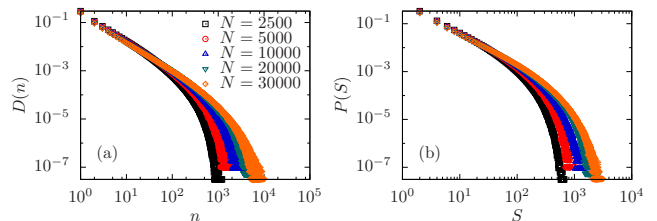


FIG. 3: (Color online) (a) Avalanche distribution $D(n)$ for the SK model recorded across the whole hysteresis loop. (b) Magnetization jump distribution $P(S)$ recorded across the whole hysteresis loop. The crossover from power-law to an exponential cutoff behavior grows noticeably with the system size signaling that in the thermodynamic limit the system displays SOC.

tion jumps are visible.

In contrast, Fig. 3 shows data for the SK model. The data are in agreement with the simulation results by Pazmandi *et al.* [5]: The distribution $D(n)$ [$P(S)$] has a power-law behavior for small n [S] with a crossover size $n^*(N)$ [$S^*(N)$] that diverges with increasing system size. A scaling collapse of the data (not shown) agrees with the estimates of Ref. [5]. Furthermore, we find that $1/n^* = 0.00011(8)$ compatible with zero for $N \rightarrow \infty$ in the SK model, i.e., $n^* = \infty$.

To further corroborate our results, in Fig. 4 we plot the crossover avalanche size n^* as a function of the space dimension d for space dimensions $d = 2$ to 8 , as well as

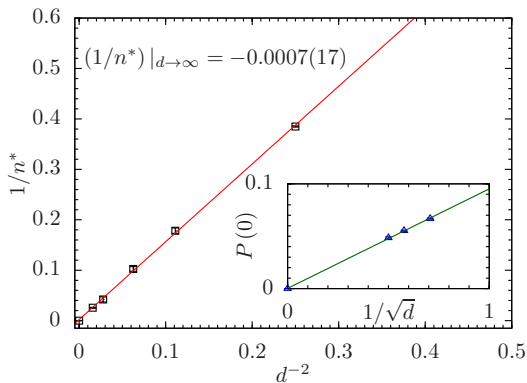


FIG. 4: (Color online) Inverse of the characteristic avalanche scale n^* as a function of space dimension d . The data extrapolate perfectly to the SK limit, meaning that for *any* finite space dimension the avalanches are finite in size. This means that SOC is a property of the infinite-ranged model only. The inset shows $P(h=0)$ as a function of $d^{-1/2}$ for different space dimensions. Again, $P(h=0) = 0$ only for the SK model (data from Ref. [27]).

$d = \infty$ (SK). The data clearly show that $n^* \propto d^2$, i.e., a true power-law behavior without a cutoff is only feasible for $d = \infty$ [25]. Note that the same behavior is obtained for avalanches restricted to $H = 0$, as well as the magnetization jumps (not shown), however with large fluctuations in the data. Finally, our results are independent of the choice of n^* (either from a fit to an exponential or from the closest distance between the fitting function), illustrating the robustness of the effect. We also compare our results to that of the pseudogap of the distribution of local fields $P(h)$, another quantity found to reflect SOC in the SK model [26]. The inset of Fig. 4 shows $P(h=0)$ as a function of the space dimension. The data clearly show that $P(h=0) \propto d^{-1/2} \rightarrow 0$ for $d \rightarrow \infty$ only.

Conclusions. — We have demonstrated that, for large system sizes, avalanches in short-range spin glasses do not span the system size, even if the space dimension d is above the upper critical dimension $d_u = 6$. This suggests that self-organized criticality as found in Ref. [5] is not necessarily a property of the mean-field regime, but is instead a result of long-range interactions. In principle, mean-field behavior can be reached in two equivalent ways: Either by increasing the space dimension above the upper critical dimension or by making the interactions infinite-ranged. Here we show that these two limits can lead to different behaviors, which become equivalent only in the $d = \infty$ and of infinite-range interaction limit. Analyzing models that allow for a continuous tuning of an effective space dimension [27–30] might thus also help in gaining further insights into this problem.

One has to keep in mind, however, that the conventional arguments [26] determining the upper critical dimension are restricted to equilibrium states – which below the glass transition temperature are typically very

difficult or impossible to reach in any experiment. In contrast, the metastable states that the system visits on the outer hysteresis loop—which our avalanches explore—are indeed as far from equilibrium as possible. One should, therefore, not naively and indiscriminately apply equilibrium concepts such as the existence of the upper critical dimension $d_u = 6$ to the far-from-equilibrium behavior we study in this letter. This might explain the disagreement with the results of Ref. [15] that suggest system-spanning avalanches for short-range spin glasses.

Our finding that the behavior of short-range models in any finite dimension remains fundamentally different than that of the infinite-range model is, therefore, a striking and a potentially far-reaching result. It calls for a change of perspective with respect to far-from-equilibrium states, and we hope that it will stimulate further serious efforts from the theoretical, but also the experimental community. The special role we suggest for the long-range interactions may have further interesting consequences, especially for bad metals near the metal-insulator transition [31]. Here the Coulomb interaction between charge carriers assumes center stage, because poor screening in the bad metal regime directly reveals its long range nature [32, 33]. Existing work has already established that the single-particle density of states, which represents the direct analogue of the $P(h)$ distribution we studied in this paper, opens a power-law “Efros-Shklovskii” gap within the Coulomb glass phase [32, 33]. Given our result that vanishing of $P(0)$ is a very direct manifestation of the SOC behavior, our findings strongly suggest that in the presence of frustrating long-range Coulomb interactions, SOC may survive [34, 35] even in physically-relevant space dimensions, such as $d = 2$ and 3. Further insights could also be gained by studying the existence of SOC for one-dimensional long-range spin glasses with power-law interactions [27–30] where an effective space dimension can be tuned.

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